The characteristic Cauchy problem for Dirac fields on curved backgrounds

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Abstract

On arbitrary spacetimes, we study the characteristic Cauchy problem for Dirac fields on a light-cone. We prove the existence and uniqueness of solutions in the future of the light-cone inside a geodesically convex neighbourhood of the vertex. This is done for data in L^2 and we give an explicit definition of the space of data on the light-cone producing a solution in H^1. The method is based on energy estimates following L. Hörmander [9].

1 Introduction

The characteristic Cauchy problem, or Goursat problem, is a Cauchy problem for a hyperbolic equation, with data set on a characteristic hypersurface. The well-posedness depends on the geometry of the characteristic hypersurface. In the typical example of the scalar wave equation on $\mathbb{R}_t \times \mathbb{R}^3_x$, specifying data on the characteristic hyperplane $t = x_1$ leads to non-unique solutions, whereas for data on the light-cone of the origin $\{t = |x|\}$, the solution exists and is unique in the future of the cone (but not in its past). In the best cases, the well-posedness will always be on one side of the hypersurface, its future or its past, unless we work on a spatially compact spacetime. A remarkable feature of the Goursat problem is that fewer data are necessary on a characteristic hypersurface than on a spacelike slice, the remaining data can be recovered by integration of the restriction of the equation to the null hypersurface.

For scalar wave equations on general globally hyperbolic curved spacetimes, the question of existence and uniqueness is well understood. A whole chapter of F.G. Friedlander's book [3] is devoted to an integral formulation of the solution for data on a light-cone using techniques due to Leray and Hadamard. Lars Hörmander [9] has proved global well-posedness for spatially compact spacetimes using a simple and natural method based on energy estimates. For spinorial zero rest-mass field equations, an integral Kirchhoff-d'Adhémard formula was obtained by R. Penrose in 1963 [12] (see also [13] Vol. 1, Section 5.11) for data on a light-cone. In the curved case, Friedlander's approach has not yet been applied to spinorial equations. However, Hörmander's method was used recently by L.J. Mason and J.-P. Nicolas [10] to construct scattering theories for Dirac and Maxwell fields via conformal methods, by J.-P. Nicolas [11] to extend Hörmander's result to metrics of weak regularity and by D. Häfner [8] to solve a Goursat problem for Dirac fields on the Kerr metric, with data on a characteristic surface generated by two congruences of outgoing and incoming null geodesics.

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The question of regularity of the solutions and its control in terms of the regularity of the data is strikingly more difficult than for the ordinary Cauchy problem, particularly so when the data is specified on a light-cone. This is simply due to the fact that the cone is a singular hypersurface. Friedlander's book gives a condition ensuring smooth solutions for scalar waves, but to our knowledge, a precise study of intermediate regularities is to this day missing. This work is a step in this direction.

We study the Goursat problem for the Dirac equation on a curved background, with data on a future light-cone. We work locally in a geodesically convex neighbourhood of the vertex, we therefore do not need to make any global hypothesis on our spacetime, such as global hyperbolicity. We find the space of data on the cone for which the problem has a unique H^1 solution. Then using density arguments, we infer a minimum regularity existence and uniqueness result. The strategy of the proof is similar to that of Häfner [8] and uses the ideas developed by Hörmander [9] for the wave equation. The case of the Dirac equation is quite different from the scalar case, not only because it is a first order hyperbolic equation, but mostly because of its spinorial nature. The control of the H^1 regularity by an adequate space of initial data requires to express the complete Dirac field on the cone in terms of the null data. This is done by solving the transport equations which are the restriction tangent to the cone of the full Dirac equation. The data for these transport equations are at the vertex and depend on the direction along which we integrate. A good understanding of this is obtained through the definition of a null tetrad, based on a choice of null coordinates in the neighbourhood of the vertex. This null tetrad is multi-valued at the vertex, but after blowing-up the tip of the cone, it is understood as a smooth frame. The regularity of the data at the tip is then controlled in terms of direction-dependent matching conditions at the blown-up vertex.

Notations. Many of our equations will be expressed using the two-component spinor notations and abstract index formalism of R. Penrose and W. Rindler [13]. Abstract indices are denoted by light face latin letters, capital for spinor indices and lower case for tensor indices. Concrete indices defining components in reference to a basis are represented by bold face latin letters. Concrete spinor indices, denoted by bold face capital latin letters, take their values in $\{0, 1\}$ while concrete tensor indices, denoted by bold face lower case latin letters, take their values in $\{0, 1, 2, 3\}$. When working with a $3 + 1$ decomposition of spacetime, we will use light face greak letters for concrete spacelike indices, taking values in $\{1, 2, 3\}$.

We will work with descriptions of the Dirac equation both in terms of Dirac spinors and Weyl (or half) spinors. For a complete account of the relations between Weyl and Dirac spinors, see [13] or [11].

2 Geometrical background

2.1 Global hyperbolicity and spin structure

In this work, we shall consider general Lorentzian manifolds and work locally in a geodesically convex neighourhood of a point. Such domains are trivially globally hyperbolic. We recall here the definition of global hyperbolicity and some of its important consequences in dimension 4, particularly regarding spinors.

A globally hyperbolic spacetime is a pair $(\mathcal{M}, g)$ where (see Geroch [7] for more details) :

- $\mathcal{M}$ is a real 4-dimensional smooth oriented, time-oriented manifold ;

- g is a smooth metric on $\mathcal{M}$ of Lorentzian signature $+ - - -$;

- there exists a global time function t on $\mathcal{M}$ such that the level hypersurfaces Σ_t of t are Cauchy hypersurfaces.

The time function t may be in addition assumed smooth (see Bernal-Sanchez [1]). Recall that a smooth time function is a smooth scalar function t on $\mathcal{M}$ such that $\nabla^a t$ is a future-oriented timelike vector field over $\mathcal{M}$; here ∇ denotes the Levi-Civita connection on $(\mathcal{M}, g)$.

Global hyperbolicity has at least two important consequences in 4 dimensions. First, the level hypersurfaces Σ_t of the time fonction t are all diffeomorphic to a given smooth 3-surface $\Sigma = \Sigma_0$ via the flow of the vector field $\nabla^a t$. Second, $\mathcal{M}$ admits a spin-structure [5, 6, 14]. We denote by $\mathbb{S}_A$ and $\bar{\mathbb{S}}_{A'}$ the bundles of left and right spinors on $\mathcal{M}$. These are 2-component spinors, or Weyl spinors. The Weyl spinor bundles are endowed with symplectic forms ε_{AB} and $\varepsilon_{A'B'}$ which are conjugates of one another. They are used to raise and lower spinor indices (meaning that they provide isomorphisms between the spin-bundles $\mathbb{S}^A$ and $\bar{\mathbb{S}}^{A'}$ and their duals $\mathbb{S}_A$ and $\bar{\mathbb{S}}_{A'}$). The bundle of Dirac spinors is defined as

$$\mathbb{S}_{Dirac} := \mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'}.$$

It is equipped with an $SL(2, \mathbb{C})$ invariant inner product expressed as

$$(\Psi\,,\,\Xi) := i\bar{\rho}_{A'}\chi^{A'} - i\phi_A\bar{\eta}^A\,, \quad \text{where } \Psi = \phi_A \oplus \chi^{A'} \text{ and } \Xi = \rho_A \oplus \eta^{A'}. \tag{1}$$

The Clifford product by any real vector is self-adjoint for the inner product (1).

The tangent bundle to $\mathcal{M}$ and the metric can be recovered from the Weyl-spinor bundles and the ε symplectic forms :

$$T\mathcal{M} \otimes \mathbb{C} = \mathbb{S}^A \otimes \bar{\mathbb{S}}^{A'}\,,$$

(a rigorous abstract index notation should in fact be $T^a\mathcal{M} \otimes \mathbb{C} = \mathbb{S}^A \otimes \bar{\mathbb{S}}^{A'}$, the vector index a corresponding to the two spinor indices A and A' clumped together), the real tangent bundle consists of the hermitian part of $\mathbb{S}^A \otimes \bar{\mathbb{S}}^{A'}$ and

$$g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}\,.$$

We can perform a $3 + 1$ decomposition of the geometry based on the time function t. We normalize the gradient of t so that its square norm equals 2 (instead of a more usual 1, this is for later convenience in the expression of the hermitian norm of spinors defined using this vector field)

$$\mathcal{T}^a := \sqrt{\frac{2}{g(\nabla t, \nabla t)}}\nabla^a t\,.$$

The metric g can then be decomposed as follows :

$$g = \frac{N^2}{2}\mathrm{d}t^2 - h(t)$$

where $-h(t)$ is the metric induced by g on Σ_t and the lapse function N is defined by

$$\mathcal{T}_a\mathrm{d}x^a = N\mathrm{d}t\,, \quad \text{or equivalently } g(\nabla t, \nabla t) = \frac{2}{N^2}\,.$$

Note that this decomposition, and more particularly the choice of product structure $\mathcal{M} = \mathbb{R}_t \times \Sigma$ associated with the integral curves of $\mathcal{T}$, fixes the meaning of the vector $\partial/\partial t$ as

$$\frac{\partial}{\partial t} = \frac{N}{2}\mathcal{T}\,.$$

Definition 2.1. *The timelike vector $\mathcal{T}$ endows the bundle of Dirac spinors with a positive definite hermitian product :*

$$\langle \Psi \,,\, \Xi \rangle := \frac{1}{\sqrt{2}} \left(\mathcal{T}.\Psi \,,\, \Xi \right) = \mathcal{T}^{AA'} \phi_A \bar{\rho}_{A'} + \mathcal{T}_{AA'} \chi^{A'} \bar{\eta}^A \,,\;\; \Psi = \phi_A \oplus \chi^{A'} \,,\;\; \Xi = \rho_A \oplus \eta^{A'} \,. \tag{2}$$

In particular we denote

$$|\Psi| = \langle \Psi, \Psi \rangle^{1/2} \,. \tag{3}$$

Remark 2.1. *Note that this definition can be naturally restricted to each of the bundles $\mathbb{S}_A$ and $\bar{\mathbb{S}}^{A'}$ and extended to $\mathbb{S}^A$ and $\bar{\mathbb{S}}_{A'}$.*

2.2 Dirac's equation, conserved quantity

We consider the charged Dirac equation associated with an electromagnetic vector-potential Φ^a, for a particle of mass m and charge q

$$\left.\begin{array}{rcl} \left(\nabla^{AA'} - iq\Phi^{AA'} \right)\phi_A & = & \frac{m}{\sqrt{2}} \chi^{A'} \,, \\[4pt] \left(\nabla_{AA'} - iq\Phi_{AA'} \right)\chi^{A'} & = & -\frac{m}{\sqrt{2}} \phi_A \,. \end{array}\right\} \tag{4}$$

This is an expression of Dirac's equation in terms of Weyl spinors, it has the form of two charged Weyl equations, one for helicity $1/2$ and the other for helicity $-1/2$, coupled by the mass. It is usual to understand Dirac's equation as an equation on the Dirac spinor $\Psi = \phi_A \oplus \chi^{A'}$. Contrary to the more elementary spin-bundles $\mathbb{S}_A$ and $\bar{\mathbb{S}}^{A'}$, the Dirac spinor bundle $\mathbb{S}_{Dirac}$ is stable under the action of the Clifford product by vectors and of the Dirac operator (see for example [11] or [13] for a description of the relations between the Dirac operator acting on Weyl spinors and on Dirac spinors)

$$\mathcal{P} \,:\, \mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'} \longrightarrow \mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'} \,,\;\; \mathcal{P} = \left(\begin{array}{cc} 0 & i\sqrt{2}\nabla_{AA'} \\ -i\sqrt{2}\nabla^{AA'} & 0 \end{array} \right) \,.$$

Equation (4) has a conserved current given by the future-oriented causal vector (see [13])

$$J^a = \phi^A \bar{\phi}^{A'} + \bar{\chi}^A \chi^{A'} \,.$$

For a given smooth spacelike or characteristic hypersurface $\mathcal{S}$, the flux of J^a through $\mathcal{S}$ is the integral over $\mathcal{S}$ of the Hodge dual $J_a \mathrm{d}^3 x^a$ of the 1-form $J_a \mathrm{d}x^a$:

$$\mathcal{E}_{\mathcal{S}} := \int_{\mathcal{S}} J_a \mathrm{d}^3 x^a, \;\; J_a \mathrm{d}^3 x^a := *J_a \mathrm{d}x^a. \tag{5}$$

The flux (5) can be understood as

$$\mathcal{E}_{\mathcal{S}} = \int_{\mathcal{S}} J_a V^a \mathrm{d}\sigma_{\mathcal{S}} \tag{6}$$

where V^a is orthogonal to $\mathcal{S}$, $d\sigma_{\mathcal{S}} = W \lrcorner \mathrm{dVol}^4$, dVol^4 being the 4-volume measure associated with the metric g and W^a a transverse vector field to $\mathcal{S}$ such that $V_a W^a = 1$. If $\mathcal{S}$ is spacelike we can take $V^a = W^a = \nu^a$, the unit future oriented normal vector field to $\mathcal{S}$. If $\mathcal{S}$ is characteristic, the vector V^a is a null vector field normal and tangent to $\mathcal{S}$ and we can chose W^a to be a null vector field transverse to $\mathcal{S}$, this provides the beginning of the construction of a Newman-Penrose tetrad (see Section 2.3).

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The $3 + 1$ decomposition of Dirac's equation reads,

$$\nabla_\mathcal{T}\Psi(t) = -\mathcal{T}.\mathcal{D}_{\Sigma_t}\Psi + k\mathcal{T}.\Psi - im\mathcal{T}.\Psi \,, \tag{7}$$

where $\mathcal{D}_{\Sigma_t}$ is the Dirac operator on $(\Sigma_t, h(t))$, $k = k(t)$ is the trace of the extrinsic curvature, in other words the mean curvature, of the slice Σ_t, divided by $\sqrt{2}$ (see [11]) and "$\mathcal{T}.$" denotes the Clifford product by the vector $\mathcal{T}$.

2.3 Newman-Penrose formalism

We will make an essential use in this work of the expression of equation (4) in the Newman-Penrose formalism. This formalism is based on the choice of a null tetrad, i.e. a set of four vector fields l^a, n^a, m^a and $\bar{m}^a$, the first two being real and future oriented, $\bar{m}^a$ being the complex conjugate of m^a, such that all four vector fields are null and m^a is orthogonal to l^a and n^a, that is to say

$$l_a l^a = n_a n^a = m_a m^a = l_a m^a = n_a m^a = 0 \,. \tag{8}$$

The tetrad is said to be normalized if in addition

$$l_a n^a = 1 \,, \; m_a \bar{m}^a = -1 \,. \tag{9}$$

To a given Newman-Penrose tetrad we can associate a spin-frame $\{o^A, \iota^A\}$, i.e. a local basis of the spin-bundle $\mathbb{S}^A$, defined uniquely up to an overall sign factor by

$$o^A \bar{o}^{A'} = l^a \,, \; \iota^A \bar{\iota}^{A'} = n^a \,, \; o^A \bar{\iota}^{A'} = m^a \,, \; \iota^A \bar{o}^{A'} = \bar{m}^a \,, \; o_A \iota^A = 1 \,. \tag{10}$$

A spin-frame $\{o^A, \iota^A\}$ satisfying $o_A \iota^A = 1$ is called unitary.

Let ϕ_0 and ϕ_1 be the components of ϕ_A in $\{o^A, \iota^A\}$, and $\chi_{0'}$ and $\chi_{1'}$ the components of $\chi_{A'}$ in $(\bar{o}^{A'}, \bar{\iota}^{A'})$:

$$\phi_0 = \phi_A o^A \,, \quad \phi_1 = \phi_A \iota^A \,, \quad \chi_{0'} = \chi_{A'} \bar{o}^{A'} \,, \quad \chi_{1'} = \chi_{A'} \bar{\iota}^{A'} \,.$$

The Dirac equation takes the form (see for example [2])

$$\left. \begin{aligned} n^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\phi_0 - m^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\phi_1 + (\mu - \gamma)\phi_0 + (\tau - \beta)\phi_1 &= \tfrac{m}{\sqrt{2}}\chi_{1'} \,, \\[2mm] l^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\phi_1 - \bar{m}^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\phi_0 + (\alpha - \pi)\phi_0 + (\varepsilon - \rho)\phi_1 &= -\tfrac{m}{\sqrt{2}}\chi_{0'} \,, \\[2mm] n^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\chi_{0'} - \bar{m}^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\chi_{1'} + (\bar{\mu} - \bar{\gamma})\chi_{0'} + (\bar{\tau} - \bar{\beta})\chi_{1'} &= \tfrac{m}{\sqrt{2}}\phi_1 \,, \\[2mm] l^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\chi_{1'} - m^\mathbf{a}(\partial_\mathbf{a} - iq\Phi_\mathbf{a})\,\chi_{0'} + (\bar{\alpha} - \bar{\pi})\chi_{0'} + (\bar{\varepsilon} - \bar{\rho})\chi_{1'} &= -\tfrac{m}{\sqrt{2}}\phi_0 \,. \end{aligned} \right\} \tag{11}$$

The μ, γ etc. are the spin coefficients which are decompositions of the connection coefficients based on the vectors of the null tetrad, for instance, $\mu = -\bar{m}^a \delta n_a$, where $\delta = m^a \nabla_a$. For the formulae of the spin coefficients and details about the Newman-Penrose formalism see [13].

A Newman-Penrose tetrad $(l, n, m, \bar{m})$ is said to be adapted to the foliation $\{\Sigma_t\}_t$ if it satisfies $l^a + n^a = \mathcal{T}^a$. The advantage of a tetrad adapted to the foliation is that the expression of the hermitian product on Dirac spinors becomes extremely simple. Let $\Psi = \phi_A \oplus \chi^{A'}$ and $\Xi = \rho_A \oplus \eta^{A'}$, denote the four components of Ψ in the spin-frame $\{o^A, \iota^A\}$ by

$$(\Psi_1, \Psi_2, \Psi_3, \Psi_4) = (\phi_0, \phi_1, \chi^{0'}, \chi^{1'}) = (\phi_0, \phi_1, \chi_{1'}, -\chi_{0'}) \,, \tag{12}$$

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then, since we have $\mathcal{T}^a = l^a + n^a = o^A \bar{o}^{A'} + \iota^A \bar{\iota}^{A'}$,

$$|\Psi|^2 = |\Psi_1|^2 + |\Psi_2|^2 + |\Psi_3|^2 + |\Psi_4|^2 \tag{13}$$

and with analogous notations for Ξ :

$$\langle \Psi, \Xi \rangle = \Psi_1 \bar{\Xi}_1 + \Psi_2 \bar{\Xi}_2 + \Psi_3 \bar{\Xi}_3 + \Psi_4 \bar{\Xi}_4 \,.$$

3   Geometrical framework

On $\mathcal{M}$, we consider a point p_0 and we work in a geodesically convex neighbourhood Ω of p_0, meaning an open subset Ω of $\mathcal{M}$ such that for any $p, q \in \Omega$, there exists a unique geodesic containing both p and q.

3.1   Important sets and coordinate system

For a point $p \in \Omega$ the future light-cone of p is defined as the set of points which are null separated from p and in the future of p

$$\mathcal{C}^+(p) := \{q \in \Omega \,;\; \text{there exists a future-oriented null geodesic from } p \text{ to } q\} \cup \{p\} \,,$$

we denote by $\mathcal{I}^+(p)$ the future chronological set of p in Ω

$$\mathcal{I}^+(p) := \{q \in \Omega \,, q \neq p \,;\; \text{there exists a future-oriented timelike geodesic from } p \text{ to } q\}$$

and by $\mathcal{J}^+(p)$ the future causal set of p in Ω

$$\mathcal{J}^+(p) = \mathcal{I}^+(p) \cup \mathcal{C}^+(p) \,.$$

We define the analogous sets in the past in the natural way : $\mathcal{C}^-(p)$, $\mathcal{I}^-(p)$ and $\mathcal{J}^-(p)$.

For the resolution of the Goursat problem, we work in a neighbourhood of the vertex of the cone $\mathcal{C}^+(p_0)$ within $\mathcal{J}^+(p_0)$, which does not need to be small, it can be large if Ω is itself large. We define this neighbourhood in two steps : first we define a closed subset D of Ω on which we construct a coordinate system, then we use this coordinate system to define the domain in which we shall solve the Goursat problem.

Definition 3.1. *Let ζ be a timelike geodesic passing through p_0. We consider a point p_1 on the curve ζ inside $\mathcal{I}^+(p_0)$ (i.e. strictly in the future of p_0). We define the domain D as*

$$D := \mathcal{J}^+(p_0) \cap \mathcal{J}^-(p_1) \,.$$

On D, we construct two foliations by light-cones which we use to define a coordinate system.

Definition 3.2. *We define a time function t on $\zeta \cap \mathcal{J}^+(p_0)$ as the metric length of ζ between p_0 and the points on $\zeta \cap \mathcal{J}^+(p_0)$. We put $T := t(p_1)/2$. For $0 \leq t \leq 2T$, we denote*

$$\mathcal{C}_t^\pm := \mathcal{C}^\pm(\zeta(t)) \cap D \,.$$

We define two null coordinates (also referred to as optical functions) u and v on D as follows :

- *$u := t$ on $\mathcal{C}_t^+$ for $0 \leq t \leq 2T$;*

- *$v := t$ on $\mathcal{C}_t^-$ for $0 \leq t \leq 2T$.*

Then we choose a coordinate system (t, r, ω) on D :

- *$t := (u + v)/2$;*

- *$r := (v - u)/2$ (or equivalently $u = t - r$ and $v = t + r$) ;*

- *$\omega \in S^2$ is first defined on the 2-surface $\mathcal{C}^+(p_0) \cap \mathcal{C}^-(p_1)$ via a choice of smooth parametrization by S^2, then its definition is extended to the whole of D by imposing that it is constant along the integral lines of ∇u and ∇v.*

Remark 3.1. *Note that the coordinate t as we have defined it on D agrees with its initial definition on ζ.*

The coordinate system (t, r, ω) has the following properties :

Proposition 3.1. *The coordinate system (t, r, ω) is smooth on $D \backslash \zeta$. The function t is a smooth time function on D. Moreover the curve ζ is an integral curve of ∂_t.*

Proof. The first statement is obvious. To prove the smoothness of t on D, it is sufficient by smooth angular dependence to prove it on a well chosen 2-surface $\mathcal{S}$ inside D. We define the surface $\mathcal{S}$ as follows. Consider a future oriented null geodesic γ_0 passing through p_0 with some affine parameter s. We parallel-transport the tangent vector to γ_0 at p_0 along the geodesic ζ. We obtain at each point $\zeta(t)$, $0 \leq t \leq 2T$, a future oriented null vector and we consider the associated null geodesic γ_t passing through $\zeta(t)$. The family of geodesics γ_t, $0 \leq t \leq 2T$, spans within D a smooth 2 surface $\mathcal{S}$. When we rotate the direction of the initial null geodesic γ_0 along all the possible directions ω (quotiented by the antipodal relation : $\omega \mapsto -\omega$), the resulting 2-surface will change smoothly and define a smooth foliation of $D \setminus \zeta$ (thanks to the geodesic convexity of D).

Now for a given geodesic γ_0, we work on the corresponding 2-surface $\mathcal{S}$. First, note that for any $t \in]0, 2T[$, the intersection $\left(\mathcal{C}_t^+ \cup \mathcal{C}_t^- \right) \cap \mathcal{S}$ is the union of two smooth curves, one being γ_t. We denote the other by β_t. The geodesic ζ splits $\mathcal{S}$ into two halves : a left and a right part. We change the definitions of u and v as follows. On the right part, we keep u and v as they are and on the left part, we exchange the roles of u and v. Then the $u = t$ curves on $\mathcal{S}$ are the γ_t's and the $v = t$ curves are the β_t's. The essential remark is that if now we put $t := u + v$, this does not change the definition of t, which shows immediately that t is smooth on $\mathcal{S}$ and thus on D.

Moreover the gradient of t is $\nabla t = \nabla u + \nabla v$ which is the sum of two future oriented null vectors on $D \backslash \zeta$. It is therefore future oriented and timelike on $D \backslash \zeta$ and thus on D by continuity.

The last property is obvious because ζ is the curve $r = 0$, $\omega = \omega_0$ for any $\omega_0 \in S^2$. $\qquad \square$

Definition 3.3. *We denote by Σ_t, $0 \leq t \leq 2T$ the level hypersurfaces of the function t in D. We shall henceforth only work in the domain*

$$D_T := D \cap \{0 \leq t \leq T\} .$$

Remark 3.2. From now on, all the sets we shall consider will be restricted to D_T. *In particular, we keep the same notations for the sets defined above, such as $\mathcal{C}^+$ for instance, but we now consider only the restriction of these sets to D_T.*

In the next subsection, we define a local frame using the construction of our coordinate system in D. Like the coordinate system, the local frame will be singular at the curve ζ : more precisely, the frame vectors will be smooth on $D_T \setminus \zeta$ and have direction dependent limits on ζ. These limits and their relations to one another will be fundamental for our constructions. A natural way to deal with them is to blow up the curve ζ as the cylinder $[0, T]_t \times S_\omega^2$. We give new notations for D_T and $\mathcal{C}_0^+$ when ζ is blown up as a cylinder :

Definition 3.4. *We shall denote by $\mathfrak{D}_T$ the domain D_T with ζ blown up, i.e. D_T considered as*

$$\{(t, r, \omega)\,;\ t \in [0, T]\,,\ r \in [0, t]\,,\ \omega \in S^2\}$$

and $\mathfrak{C}$ the cone $\mathcal{C}_0^+$ considered as

$$[0, 2T]_v \times S_\omega^2\,.$$

3.2 Newman-Penrose tetrad

We now construct on D_T a Newman-Penrose tetrad which is adapted to the foliation. The vector $\mathcal{T}^a$, future oriented, normal to the foliation $\{\Sigma_t\}_{0 \leq t \leq T}$ of D_T is given by

$$\mathcal{T}^a \partial_a = \sqrt{\frac{2}{g(\nabla t, \nabla t)}} \nabla t = N \nabla t\,.$$

Definition 3.5. *We define at each point of $D_T \setminus \zeta$ the vectors l and n of our Newman-Penrose tetrad by*

$$l = \frac{\nabla u}{\sqrt{2\,g(\nabla t, \nabla t)}} = \frac{N}{2} \nabla u\,,\ n = \frac{\nabla v}{\sqrt{2\,g(\nabla t, \nabla t)}} = \frac{N}{2} \nabla v\,.$$

This defines l and n as smooth future null vector fields on $D_T \setminus \zeta$, l being tangent (and orthogonal) to the cones $\mathcal{C}_t^+$ and n to the cones $\mathcal{C}_t^-$. A choice of l and n fixes the vector m uniquely up to a complex factor of modulus 1. We make a choice of m so that it is a smooth vector field on $\mathfrak{D}_T$. The vectors m and $\bar{m}$ are by construction tangent to the 2-surfaces where both u and v (equivalently both t and r) are constant, i.e. the intersections $\mathcal{C}_u^+ \cap \mathcal{C}_v^-$.

We then extend this Newman-Penrose tetrad as a smooth Newman-Penrose tetrad on $\mathfrak{D}_T$: we define a Newman-Penrose tetrad at each point $\zeta(t)$ for each direction ω by taking the limit of $(l, n, m, \bar{m})$ along the integral curve of ∇u on $\mathcal{C}_t^+$ corresponding to the direction ω.

Remark 3.3. *It is important to note that by continuity, the tetrads defined on ζ are still adapted to the foliation, i.e. they satisfy $l^a + n^a = \mathcal{T}^a$.*

It is useful to calculate explicitly the coordinate vector fields ∂_u and ∂_v :

Lemma 3.1.

$$\begin{aligned}
\frac{\partial}{\partial u} &= \frac{N^2}{4} \nabla v = \frac{N}{2}\, n\,, \\
\frac{\partial}{\partial v} &= \frac{N^2}{4} \nabla u = \frac{N}{2}\, l\,.
\end{aligned}$$

Proof. By construction, the vectors ∂_u and ∂_v are in the plane spanned by ∇u and ∇v. They can therefore be decomposed along the vectors l and n as follows :

$$\partial_u = g(\partial_u, l)n + g(\partial_u, n)l\,,\ \partial_v = g(\partial_v, l)n + g(\partial_v, n)l\,.$$

We perform the calculation explicitly for ∂_u :

$$\begin{aligned}
\partial_u &= g(\partial_u, l)n + g(\partial_u, n)l \\
&= \frac{N^2}{4}\,(g(\partial_u, \nabla u)\nabla v + g(\partial_u, \nabla v)\nabla u) \\
&= \frac{N^2}{4}\,(\mathrm{d}u(\partial_u)\nabla v + \mathrm{d}v(\partial_u)\nabla u) \\
&= \frac{N^2}{4} \nabla v\,.
\end{aligned}$$

The calculation for ∂_v is similar. $\qquad\square$

3.3 Structure at the tip of the cone viewed on $\mathfrak{C}$

At p_0 (just as at any other point of the timelike curve ζ), the Newman-Penrose tetrad which we have defined is singular and multi-valued. We denote by $\gamma_{0,\omega}$ the integral curve of ∇u on $\mathcal{C}_0^+$ corresponding to the direction ω and by $(l_\omega, n_\omega, m_\omega, \bar{m}_\omega)$ the Newman-Penrose tetrad $(l, n, m, \bar{m})$ at p_0 corresponding to the direction ω. Let $\{o_\omega^A, \iota_\omega^A\}$ be the associated spin-frame. The vector l_ω points in a direction corresponding to ω and the vector n_ω points along another direction on S^2. This direction will be important when we express in terms of components the restriction of the Dirac equation to the null curve $\gamma_{0,\omega}$.

Definition 3.6. *We denote by ω' the direction on S^2 corresponding to n_ω and call it the conjugate direction of ω.*

The directions ω and ω' and their associated tetrads satisfy some important properties.

Lemma 3.2. *Given any direction $\omega \in S^2$ and ω' its conjugate direction, we have :*

- $(\omega')' = \omega$;

- *the relation between the null tetrads $\{l_\omega, n_\omega, m_\omega, \bar{m}_\omega\}$ and $\{l_{\omega'}, n_{\omega'}, m_{\omega'}, \bar{m}_{\omega'}\}$ is given by*

$$l_{\omega'} = n_\omega , \ n_{\omega'} = l_\omega , \ m_{\omega'} = e^{i\theta(\omega)}\bar{m}_\omega , \ \bar{m}_{\omega'} = e^{-i\theta(\omega)}m_\omega ,$$

  where $\theta(\omega) \in \mathbb{R}/2\pi\mathbb{Z}$ is a function of ω which is smooth on S^2 and such that

$$\theta(\omega) = \theta(\omega') .$$

Proof. The property that $(\omega')' = \omega$ follows from

$$\mathcal{T}^a(p_0) = l_\omega^a + n_\omega^a = l_{\omega'}^a + n_{\omega'}^a . \tag{14}$$

Indeed, any plane in the tangent space to p_0 contains at most two null directions. The plane spanned by l_ω^a and n_ω^a contains exactly two which are precisely l_ω^a and n_ω^a. This plane also contains $\mathcal{T}^a(p_0)$. Now by definition $l_{\omega'}$ is colinear to n_ω and $n_{\omega'} = \mathcal{T}^a(p_0) - l_{\omega'}$ is a null direction in this plane which is distinct from that of $l_{\omega'}$. Hence $n_{\omega'}$ must be colinear to l_ω. Using (14) again, we get $l_{\omega'}^a = n_\omega^a$ and $n_{\omega'}^a = l_\omega^a$. This implies that the spin-frame is transformed as

$$o_{\omega'}^A = \alpha \iota_\omega^A , \ \iota_{\omega'}^A = \beta o_\omega^A$$

where $\alpha, \beta \in \mathbb{C}$ and satisfy $|\alpha| = |\beta| = 1$. Then,

$$m_{\omega'}^a = o_{\omega'}^A \bar{\iota}_{\omega'}^{A'} = \alpha\bar{\beta}\iota_\omega^A \bar{o}_\omega^{A'} = \alpha\bar{\beta}\bar{m}_\omega^a ,$$

where $|\alpha\bar{\beta}| = 1$. Putting $e^{i\theta} = \alpha\bar{\beta}$, the proof is complete. The other properties of θ follow by construction. $\square$

We can establish a similar relation for the spin-frame modulo an overall sign choice.

Lemma 3.3. *Given any direction $\omega \in S^2$ and ω' its conjugate direction, the relation between the dyads $\{o_\omega^A, \iota_\omega^A\}$ and $\{o_{\omega'}^A, \iota_{\omega'}^A\}$ is given by (after a choice of overall sign)*

$$o_{\omega'}^A = ie^{i\theta(\omega)/2}\iota_\omega^A , \ \iota_{\omega'}^A = ie^{-i\theta(\omega)/2}o_\omega^A .$$

For the conjugate spin-frames, we have consequently

$$\bar{o}_{\omega'}^{A'} = -ie^{-i\theta(\omega)/2}\bar{\iota}_\omega^{A'} , \ \bar{\iota}_{\omega'}^{A'} = -ie^{i\theta(\omega)/2}\bar{o}_\omega^{A'} .$$

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Proof. We continue from the proof of Lemma 3.2. We must have

$$1 = \varepsilon_{AB} o^A_{\omega'} \iota^B_{\omega'} = \alpha\beta \varepsilon_{AB} \iota^A_\omega o^B_\omega = -\alpha\beta$$

and since $|\alpha| = |\beta| = 1$, we must have

$$\beta = -\frac{1}{\alpha} = -\bar{\alpha} \,.$$

Therefore, $e^{i\theta} = \alpha\bar{\beta} = -\alpha^2$. Our choice of sign corresponds to $\alpha = ie^{i\theta/2}$. $\qquad\square$

This entails the following relations between the components of a Dirac spinor in the spin-frames $\{o^A_\omega \,, \ \iota^A_\omega\}$ and $\{o^A_{\omega'} \,, \ \iota^A_{\omega'}\}$:

Corollary 3.1. *Let* $\Psi = \phi_A \oplus \chi^{A'}$ *be a Dirac spinor at p_0, denote by $\Psi_i(\omega)$, $i = 1, 2, 3, 4$ its components in the spin-frame $\{o^A_\omega \,, \ \iota^A_\omega\}$, defined by (12). We have*

$$\Psi_2(\omega') = ie^{-i\theta(\omega)/2}\Psi_1(\omega) \,, \ \ \Psi_3(\omega') = ie^{i\theta(\omega)/2}\Psi_4(\omega) \,,$$
$$\Psi_1(\omega') = ie^{i\theta(\omega)/2}\Psi_2(\omega) \,, \ \ \Psi_4(\omega') = ie^{-i\theta(\omega)/2}\Psi_3(\omega) \,.$$

Remark 3.4. *Applying this transformation twice leads to a global sign change of the components of Ψ. This is a typical consequence of the fact that the bundle of unitary spinor dyads is a two-fold covering of the bundle of normalized Newman-Penrose tetrads.*

3.4 Some function spaces

We first define the space $\mathcal{F}$ of "smooth" characteristic data on $\mathcal{C}^+_0$. All other function spaces on the cone will be defined as completions of $\mathcal{F}$ in a given norm. Of course spinor fields on $\mathcal{C}^+_0$ cannot be smooth since $\mathcal{C}^+_0$ is not a smooth hypersurface ; the idea is to define a space of characteristic data on $\mathcal{C}^+_0$ for equation (4) which are as smooth as the cone will allow. For this, we pull smooth spinor fields on Σ_T back onto $\mathcal{C}^+_0$ using the flow of ∂_t and then we keep only the part of the resulting spinors that is transverse to $\mathcal{C}^+_0$. This can be made quite explicit with a parametrization of $\mathcal{C}^+_0$ based on Σ_T.

Definition 3.7. *Let $f : \Sigma_T \longrightarrow \mathcal{C}^+_0$ defined in our coordinate system as*

$$f(r, \omega) := (t = r, r, \omega) \,.$$

In other words, we identify points on $\mathcal{C}^+_0$ with points on Σ_T via the flow of the vector field ∂_t.

This parametrization is a Lipschitz diffeomorphism from Σ_T onto the cone $\mathcal{C}^+_0$ and it is a $\mathcal{C}^\infty$-diffeomorphism from $\Sigma_T \setminus \{\zeta(T)\}$ onto $\mathcal{C}^+_0 \setminus \{p_0\}$.

Definition 3.8. *We define the space F of "smooth" Dirac spinors on $\mathcal{C}^+_0$ as the set of Dirac spinor fields Ψ on $\mathcal{C}^+_0$ defined by a spinor field $\Xi \in \mathcal{C}^\infty(\Sigma_T \,; \ \mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ as follows*

$$\Psi(r, r, \omega) = \left(\Phi_{\partial_t}(T - r)\right)^* \left(\Xi(T, r, \omega)\right) \,.$$

Then we keep the part of the elements of F that is transverse to $\mathcal{C}^+_0$:

$$\mathcal{F} := \{(\Psi_1, \Psi_4) \,, \ where \ \Psi \in F\} \,.$$

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Remark 3.5. *Any 2-spinor ϕ_A at a point of $\mathcal{C}_0^+$ can be decomposed as*

$$\phi_A = \phi_1 o_A - \phi_0 \iota_A\,, \;\; \phi_0 = \phi_A o^A\,, \;\; \phi_1 = \phi_A \iota^A\,.$$

The spinor o^A points along l^a (in the sense of its flag-pole direction, see [13], Vol. 1) which is tangent to $\mathcal{C}_0^+$, whereas ι^A points along n^a which is trasverse to $\mathcal{C}_0^+$. So we can consider ϕ_0 as the part of ϕ_A transverse to $\mathcal{C}_0^+$ and ϕ_1 as the part of ϕ_A tangent to $\mathcal{C}_0^+$.

Remark 3.6. *Any local diffeomorphism of the tangent bundle induces a local diffeomorphism of the spin-bundle modulo a choice of sign, which means the choice of a sheat in a two-fold covering. For the flow of a vector field, the choice of sign is globally imposed by continuity and the natural requirement that for $t = 0$, the push-forward and the pull-back are the identity. Hence there is no ambiguity in the meaning of $(\Phi_{\partial_t}(T-r))^*$ applied to a spinor at the point (T, r, ω).*

The first function space we define as a completion of $\mathcal{F}$ is the space of L^2 characteristic data :

Definition 3.9. *Let the space $L^2((\mathcal{C}_0^+, d\sigma_{\mathcal{C}_0^+}); \mathbb{C}^2)$ be the completion of $\mathcal{F}$ in the norm*

$$\|(\Psi_1, \Psi_4)\|_{L^2(\mathcal{C}_0^+)}^2 := \int_{\mathcal{C}_0^+} (|\Psi_1|^2 + |\Psi_4|^2) d\sigma_{\mathcal{C}_0^+}\,,$$

where $d\sigma_{\mathcal{C}_0^+} = n \lrcorner dVol^4$ and $dVol^4$ is the 4-volume measure induced by g.

Remark 3.7. *An important property of our coordinate system is that the singularity of the diffeomorphism f at $r = 0$ is explicitly purely radial. The vectors m and $\bar{m}$ being tangent to the 2-surfaces of constant (t, r), it follows that the differential operators $m^a \partial_a$ and $\bar{m}^a \partial_a$ at any given power send $\mathcal{F}$ into $L^2((\mathcal{C}_0^+\,;\; d\sigma_{\mathcal{C}_0^+})\,;\; \mathbb{C}^2)$.*

4   Results

4.1   The L^2 setting

Let $\Psi_T \in \mathcal{C}^\infty(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$. By the usual theorems for hyperbolic equations there exists a unique solution $\Psi = \phi_A \oplus \chi^{A'} \in \mathcal{C}^\infty(D_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$ of (4) such that the trace of Ψ on Σ_T is equal to Ψ_T (see [11] for details). We can introduce the linear trace operator :

Definition 4.1. *Let Γ be the operator which, to smooth data $\Psi_T \in \mathcal{C}^\infty(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$, associates the pair of complex scalar functions (Ψ_1, Ψ_4) on $\mathfrak{C}$, first and fourth components in the spin-frame (o^A, ι^A) of the restriction of the corresponding solution Ψ to the cone $\mathcal{C}_0^+$. By construction, we have*

$$\Gamma : \Psi_T \in \mathcal{C}^\infty(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'}) \mapsto (\Psi_1, \Psi_4) \in L^2((\mathcal{C}_0^+, d\sigma_{\mathcal{C}_0^+}); \mathbb{C}^2)\,,$$

since (Ψ_1, Ψ_4) are in fact smooth scalar functions on $\mathfrak{C}$.

Using the conserved current we obtain by Stokes' theorem :

$$\int_{\Sigma_T} *(\phi_A \bar{\phi}_{A'} dx^{AA'} + \bar{\chi}_A \chi_{A'} dx^{AA'}) = \int_{\mathcal{C}_0^+} *(\phi_A \bar{\phi}_{A'} dx^{AA'} + \bar{\chi}_A \chi_{A'} dx^{AA'})$$

which can be written explicitly in terms of components of Ψ as

$$\int_{\Sigma_T} |\Psi|^2 \, d\sigma_{\Sigma_T} = \int_{\mathcal{C}_0^+} (|\Psi_1|^2 + |\Psi_4|^2) d\sigma_{\mathcal{C}_0^+}, \tag{15}$$

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where $d\sigma_{\Sigma_T} = \frac{1}{2}\mathcal{T}\lrcorner d\text{Vol}^4$.

Equation (15) entails that the operator Γ possesses an extension to a bounded operator

$$\Gamma \in \mathcal{L}\left(L^2\left(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'}\right); L^2\left((\mathcal{C}_0^+, d\sigma_{\mathcal{C}_0^+}); \mathbb{C}^2\right)\right). \tag{16}$$

Our first result is

Theorem 1. *The operator Γ is an isometry.*

4.2 Further L^2 estimates

In this subsection we consider the Dirac equation with a source $\Xi = \rho_A \oplus \eta^{A'}$:

$$\begin{cases} \left(\nabla^{AA'} - iq\Phi^{AA'}\right)\phi_A &= \frac{m}{\sqrt{2}}\chi^{A'} + \eta^{A'}, \\ \left(\nabla_{AA'} - iq\Phi_{AA'}\right)\chi^{A'} &= -\frac{m}{\sqrt{2}}\phi_A + \rho_A, \end{cases} \tag{17}$$

We have :

Lemma 4.1. *Let $\Xi = \rho_A \oplus \eta^{A'} \in \mathcal{C}^\infty(D_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$ and $\Psi = \phi_A \oplus \chi^{A'} \in \mathcal{C}^\infty(D_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$ be a smooth solution of (17). Then :*

$$\left|\int_{\Sigma_T} |\Psi|^2 \, d\sigma_{\Sigma_T} - \int_{\mathcal{C}_0^+} (|\Psi_1|^2 + |\Psi_4|^2) d\sigma_{\mathcal{C}_0^+}\right| \lesssim \int_0^T \int_{\Sigma_t} (|\Psi|^2 + |\Xi|^2) d\sigma_{\Sigma_t} dt \tag{18}$$

and for $0 < t < T$,

$$\int_{\Sigma_t} |\Psi|^2 \, d\sigma_{\Sigma_t} \lesssim \int_{\Sigma_T} |\Psi|^2 \, d\sigma_{\Sigma_T} + \int_t^T \int_{\Sigma_s} |\Xi|^2 \, d\sigma_{\Sigma_s} ds. \tag{19}$$

Proof. We shall use the following notations for $0 \leq t_1 < t_2 \leq T$,

$$\mathcal{S}_{t_1,t_2} := \mathcal{C}_0^+ \cap \{t_1 \leq t \leq t_2\}, \quad D_{t_1,t_2} := D_T \cap \{t_1 \leq t \leq t_2\}. \tag{20}$$

It is sufficient to establish the two estimates for the Weyl equation :

$$\nabla^{AA'}\phi_A = \eta^{A'} \tag{21}$$

with $\eta^{A'} \in \mathcal{C}^\infty(D_T; \mathbb{S}^{A'})$. We apply Stokes' theorem on the closed hypersurface made of Σ_T, Σ_t and the 3-surface $\mathcal{S}_{t,T}$. We obtain :

$$\int_{\Sigma_T} *(\phi_A \bar{\phi}_{A'} dx^{AA'}) - \int_{\Sigma_t} *(\phi_A \bar{\phi}_{A'} dx^{AA'}) = \int_{\mathcal{S}_{t,T}} |\phi_0|^2 d\sigma_{\mathcal{C}_0^+} + 2\Re \int_{D_{t,T}} \eta^{A'}\bar{\phi}_{A'} d\text{Vol}^4. \tag{22}$$

We have :

$$\left|\int_{D_{t,T}} \eta^{A'}\bar{\phi}_{A'} d\text{Vol}^4\right| \lesssim \int_t^T \int_{\Sigma_s} |\eta^{A'}\bar{\phi}_{A'}| d\sigma_{\Sigma_s} ds \lesssim \int_t^T \int_{\Sigma_s} (T^{AA'}\phi_A\bar{\phi}_{A'} + T^{AA'}\bar{\eta}_A\eta_{A'}) d\sigma_{\Sigma_s} ds.$$

This entails (18) putting $t = 0$ and (19) via a Gronwall estimate. $\qquad\square$

4.3 Transport equations along the cone

The image of the data for (4) by the trace operator Γ only involves two components of the trace of the solution on $\mathcal{C}_0^+$: $\Psi_1 = \phi_0$ and $\Psi_4 = -\chi_{0'}$. The other two are completely determined by the value of the solution at the vertex of the cone and by the restriction of the Dirac equation tangent to $\mathcal{C}_0^+$:

$$\left. \begin{aligned} l^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\phi_1 - \bar{m}^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\phi_0 + (\alpha - \pi)\phi_0 + (\varepsilon - \rho)\phi_1 = -\tfrac{m}{\sqrt{2}}\chi_{0'}\,, \\[2mm] l^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\chi_{1'} - m^{\mathbf{a}}(\partial_{\mathbf{a}} - iq\Phi_{\mathbf{a}})\,\chi_{0'} + (\bar{\alpha} - \bar{\pi})\chi_{0'} + (\bar{\varepsilon} - \bar{\rho})\chi_{1'} = -\tfrac{m}{\sqrt{2}}\phi_0\,. \end{aligned} \right\} \tag{23}$$

On $\mathcal{C}_0^+$ we use the coordinate system (v,ω) which realizes the cone as $\mathfrak{C}$ with its vertex blown up as a 2-sphere. Using the relation between l and ∂_v given in Lemma 3.1, the equations (23) can be written as

$$\partial_v \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} = \frac{N}{2} P_\omega \begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} + Q_1 \begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} + Q_2 \begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix}, \tag{24}$$

where

$$P_\omega = \begin{pmatrix} \bar{m}^a \partial_a & 0 \\ 0 & -m^a \partial_a \end{pmatrix}.$$

Neither the operator P_ω nor the potentials Q_1 and Q_2 are smooth at the vertex of the cone but they are smooth on $\mathfrak{C}$. The initial condition is (see Corollary 3.1) :

$$\begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix}(0,\omega) = \begin{pmatrix} -ie^{-i\theta(\omega)/2}\Psi_1 \\ -ie^{i\theta(\omega)/2}\Psi_4 \end{pmatrix}(0,\omega').$$

We find the solution :

$$\begin{aligned}
\begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix}(v,\omega) &= e^{\int_0^v Q_2(v',\omega)\mathrm{d}v'}\left[\begin{pmatrix} -ie^{-i\theta(\omega)/2}\Psi_1 \\ -ie^{i\theta(\omega)/2}\Psi_4 \end{pmatrix}(0,\omega') \right. \\
&\qquad \left. + \int_0^v e^{-\int_0^{v'} Q_2(s,\omega)\mathrm{d}s}\left(\frac{N}{2}P_\omega + Q_1\right)\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix}(v',\omega)\mathrm{d}v' \right] \\
&=: \left(K\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix}\right)(v,\omega). \tag{25}
\end{aligned}$$

4.4 The H^1 setting

In this subsection, we work entirely with the component version of Ψ. We still denote by Ψ the vector whose components are $\Psi_1, ..., \Psi_4$. The covariant derivative $\nabla_T \Psi$ expressed in terms of the components of Ψ is equal to $T^a \partial_a \Psi$ (denoted $T\Psi$) plus a matrix of connection coefficients applied to Ψ. Equation (7) can be written as

$$T\Psi = iH\Psi \tag{26}$$

where iH contains the right hand-side of (7) with the additional connection terms mentioned above coming from the time derivative $\nabla_T \Psi$. By standard theorems for Dirac operators on Riemannian manifolds, the operator $\slashed{D}_{\Sigma_t}$ is elliptic (see for example [4]), hence the norm

$$\|\Psi(t)\|_{H^1}^2 := \|H\Psi\|_{L^2(\Sigma_t)}^2 + \|\Psi\|_{L^2(\Sigma_t)}^2$$

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is equivalent to the natural H^1 norm on Σ_t uniformly in $t \in]0, T]$.

If Ψ satisfies equation (26), then $\mathcal{T}\Psi$ satisfies

$$\mathcal{T}(\mathcal{T}\Psi) = iH(\mathcal{T}\Psi) + [\mathcal{T}, iH]\Psi \tag{27}$$

and the error term $[\mathcal{T}, iH]\Psi$ is controlled in norm by $\|\Psi\|_{H^1}$. From lemma 4.1, $\mathcal{T}\Psi$ satisfies the estimates

$$\left| \int_{\Sigma_T} |\mathcal{T}\Psi|^2 \, d\sigma_{\Sigma_T} - \int_{\mathcal{C}_0^+} (|\mathcal{T}\Psi_1|^2 + |\mathcal{T}\Psi_4|^2) d\sigma_{\mathcal{C}_0^+} \right| \lesssim \int_0^T \|\Psi(t)\|_{H^1}^2 \, dt \tag{28}$$

and for $t \in [0, T]$,

$$\int_{\Sigma_t} |\mathcal{T}\Psi|^2 \, d\sigma_{\Sigma_t} \lesssim \int_{\Sigma_T} |\mathcal{T}\Psi|^2 \, d\sigma_{\Sigma_T} + \int_t^T \int_{\Sigma_s} \|\Psi\|_{H^1}^2 \, d\sigma_{\Sigma_s} ds. \tag{29}$$

On the cone we have, using (11)

$$
\begin{aligned}
\mathcal{T}\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} &= (l+n)\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} \\
&= l\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} + \overline{P_\omega}\begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} + Q_3\begin{pmatrix} \Psi_2 \\ \Psi_3 \end{pmatrix} + Q_4\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} \\
&= \left(l + Q_4 + (\overline{P_\omega} + Q_3)K\right)\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} \\
&=: L\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix},
\end{aligned}
$$

where Q_3 and Q_4 are smooth potentials on $\mathfrak{C}$.

The operator L is well defined as an operator from $\mathcal{F}$ to $L^2((\mathcal{C}_0^+; \, d\sigma_{\mathcal{C}_0^+}); \, \mathbb{C}^2)$. We now define on $\mathcal{C}_0^+$ the Hilbert space

Definition 4.2. *Let* $\mathcal{H}_{\mathcal{C}_0^+}$ *be the completion of* $\mathcal{F}$ *in the norm*

$$\left\| \begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} \right\|_{\mathcal{H}_{\mathcal{C}_0^+}}^2 := \left\| \begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} \right\|_{L^2((\mathfrak{C}; \, d\sigma_{\mathcal{C}_0^+}); \, \mathbb{C}^2)}^2 + \left\| L\begin{pmatrix} \Psi_1 \\ \Psi_4 \end{pmatrix} \right\|_{L^2((\mathfrak{C}; \, d\sigma_{\mathcal{C}_0^+}); \, \mathbb{C}^2)}^2 .$$

Using (28), (29) and a Gronwall estimate, we get

Lemma 4.2. *For all smooth data* $\Psi_T \in \mathcal{C}^\infty(\Sigma_T; \mathbb{S}_A \oplus \mathbb{S}^{A'})$, *we have*

$$\|\Psi_T\|_{H^1(\Sigma_T)} \lesssim \|\Gamma\Psi_T\|_{\mathcal{H}_{\mathcal{C}_0^+}} \lesssim \|\Psi_T\|_{H^1(\Sigma_T)} \tag{30}$$

and the trace operator Γ *therefore extends as a continuous operator from* $H^1(\Sigma_T)$ *to* $\mathcal{H}_{\mathcal{C}_0^+}$.

The main result of this paper, of which theorem 1 is a consequence, is :

Theorem 2. Γ *is an isomorphism from* $H^1(\Sigma_T)$ *onto* $\mathcal{H}_{\mathcal{C}_0^+}$.

4.5 The Cauchy problem on a rough hypersurface on spatially compact space-times

This section contains an extension to the Dirac equation in 4 spacetime dimensions, of the results of [9] for the Cauchy problem on a Lipschitz hypersurface. We consider a smooth compact manifold X without boundary of dimension 3. The spacetime $\mathcal{X} := \mathbb{R}_t \times X$ is endowed with a smooth Lorentzian metric g of the form

$$g = \frac{N^2}{2} \mathrm{d}t^2 - h(t)$$

where $h(t)$ is a time-dependent Riemannian metric on X. We denote as before

$$\mathcal{T} := \frac{2}{N} \frac{\partial}{\partial t} \,.$$

By [5, 6, 14], $\mathcal{X}$ admits a spin structure. We denote by $|\Psi|$ the norm induced by $\mathcal{T}$ on Dirac spinors at a point. Let X_t denote the hypersurface $\{t\} \times X$ for any $t \in \mathbb{R}$. Using the parallelizability of X, we consider on $\mathcal{X}$ a global smooth orthonormal frame

$$e_0 = \frac{1}{\sqrt{2}} \mathcal{T} \,, \ e_\alpha \,, \ \alpha = 1, 2, 3 \,.$$

We also introduce a smooth density $\mathrm{d}\mu$ on X, for example the one induced by the metric $h(0)$. We define two spaces of Dirac spinor fields on X :

Definition 4.3. *The spaces $L^2(X \,;\, \mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ and $H^1(X \,;\, \mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ are the completions of the space of smooth Dirac spinor fields on X in the norms*

$$\|\Psi\|^2_{L^2(X)} := \int_X |\Psi(x)|^2 \, \mathrm{d}\mu \,,$$

$$\|\Psi\|^2_{H^1(X)} := \|\Psi\|^2_{L^2(X)} + \sum_{\alpha=1}^3 \|\nabla_{e_\alpha} \Psi\|^2_{L^2(X)} \,.$$

When we need to use the L^2 norm induced by the conserved quantity for (4) *on a specific X_t, we shall denote*

$$\|\Psi\|_{L^2(X_t)} := \int_{X_t} \mathcal{T}^a \left(\phi_A \bar{\phi}_{A'} + \chi_{A'} \bar{\chi}_A \right) \mathrm{d}\sigma_{X_t} \,, \ \mathrm{d}\sigma_{X_t} = \frac{1}{2} \mathcal{T} \lrcorner \mathrm{dVol}^4|_{X_t} \,.$$

Let $\mathcal{S}$ be a Cauchy hypersurface in $\mathcal{X}$ with low regularity, defined as the graph of a function $f : X \to \mathbb{R}$ which is merely assumed Lipschitz-continuous on X. Lipschitz continuous functions are differentiable almost everywhere, hence, the normal vector field V^a to $\mathcal{S}$, defined by

$$V = e_0 + \sum_{\alpha=1}^3 \nabla_{e_\alpha} f \, e_\alpha \,, \tag{31}$$

and the tangent vectors to $\mathcal{S}$

$$\tau_\alpha = \nabla_{e_\alpha} f \, e_0 + e_\alpha \,,$$

are defined almost everywhere on $\mathcal{S}$ and are in $L^\infty(\mathcal{S})$. We assume that $\mathcal{S}$ is uniformly spacelike, i.e. there exists $0 < \varepsilon < 1$ such that, almost everywhere on X, $g_{ab} V^a V^b \geq \varepsilon$, or equivalently,

$$\sum_{\alpha=1}^3 (\nabla_{e_\alpha} f)^2 \leq 1 - \varepsilon \,. \tag{32}$$

The hypersurface $\mathcal{S}$ being Lipschitz allows us to define on $\mathcal{S}$ Sobolev spaces H^s for $s = 0, 1$:

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Definition 4.4. *For $s = 0, 1$, we define the space $H^s(\mathcal{S}\,;\,\mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ as the set of Dirac spinor fields Ψ on $\mathcal{S}$ defined by a spinor field $\Xi \in H^s(X_0\,;\,\mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ as follows*

$$\Psi(f(x), x) = \left(\Phi_{\partial_t}(-f(x))\right)^* \left(\Xi(0, x)\right) .$$

On $H^0(\mathcal{S}\,;\,\mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'}) = L^2(\mathcal{S}\,;\,\mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$, we shall use the norm induced by the conserved quantity for (4) :

$$
\begin{aligned}
\|\Psi\|^2_{L^2(\mathcal{S})} \;=\; \int_X \Big(& \nu^{AA'}(f(x), x)\phi_A(f(x), x)\bar{\phi}_{A'}(f(x), x) \\
& + \nu_{AA'}(f(x), x)\chi^{A'}(f(x), x)\bar{\chi}^A(f(x), x) \Big)\, \mathrm{d}\sigma(f(x), x) ,
\end{aligned}
\tag{33}
$$

where $\mathrm{d}\sigma$ is the Leray form on $\mathcal{S}$ associated with the parametrization of $\mathcal{S}$ by f, i.e.

$$\mathrm{d}\sigma = \nu \lrcorner \mathrm{dVol}^4 ,$$

and ν^a is the future oriented unit normal vector field to $\mathcal{S}$

$$\nu^a = \frac{V^a}{g_{bc}V^b V^c} .$$

The space $H^1(\mathcal{S}\,;\,\mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ can be understood as the set of Dirac spinor fields Ψ defined on $\mathcal{S}$ such that Ψ and its tangential derivatives $\nabla_{\tau_\alpha}\Psi$ are in $L^2(\mathcal{S}\,;\,\mathbb{S}_A \oplus \bar{\mathbb{S}}^{A'})$ and we put

$$\|\Psi\|^2_{H^1(\mathcal{S})} := \|\Psi\|^2_{L^2(\mathcal{S})} + \sum_{\alpha=1}^{3} \|\nabla_{\tau_\alpha}\Psi\|^2_{L^2(\mathcal{S})} .$$

We have the following theorem :

Theorem 3. *Let $\Phi \in L^2(\mathcal{S})$, there exists a unique solution*

$$\Psi \in \mathcal{C}\left(\mathbb{R}_t\,;\,L^2(X)\right)$$

of (4) such that

$$\Psi|_{\mathcal{S}} = \Phi .$$

Moreover, if $\Phi \in H^1(\mathcal{S})$, then

$$\Psi \in \mathcal{C}\left(\mathbb{R}_t\,;\,H^1(X)\right) \cap \mathcal{C}^1\left(\mathbb{R}_t\,;\,L^2(X)\right) .$$

5 Proofs of the main results

5.1 Proof of Theorem 3

First we establish some L^2 and H^1 energy estimates between X_0 and $\mathcal{S}$. Let us denote

$$T_1 := \min_{x \in X} f(x) , \quad T_2 := \max_{x \in X} f(x) .$$

Lemma 5.1. *For any smooth solution Ψ of (4) on $\mathcal{X}$,*

$$\|\Psi\|^2_{L^2(\mathcal{S})} = \|\Psi\|^2_{L^2(X_0)} .
\tag{34}$$

Moreover, there exist constants $0 < C_1 < C_2 < +\infty$ independent of Ψ and depending only on T_1, T_2 and the Lipschitz norm of f on X, such that

$$C_1 \|\Psi\|^2_{H^1(\mathcal{S})} \leq \|\Psi\|^2_{H^1(X_0)} \leq C_2 \|\Psi\|^2_{H^1(\mathcal{S})} .
\tag{35}$$

Remark 5.1. *Using the conserved current, we immediately obtain that for any smooth solution Ψ of (4) on $\mathcal{X}$,*

$$\|\Psi\|^2_{L^2(X_{t_1})} = \|\Psi\|^2_{L^2(X_{t_2})}$$

for any $t_1, t_2 \in \mathbb{R}$. So for equality (34) it does not matter whether we choose X_0 or any other slice X_t.

Proof of Lemma 5.1. We first prove (34). The equality is trivial for smooth hypersurfaces $\mathcal{S}$ using the conserved current for equation (4). We consider a sequence of smooth hypersurfaces $\mathcal{S}_n$ approaching $\mathcal{S}$ as follows[3] : each $\mathcal{S}_n$ is defined as the graph of a smooth function $f_n : X \to \mathbb{R}$, $f_n \to f$ in $L^\infty(X)$, $\partial_\alpha f_n \to \partial_\alpha f$ almost everywhere on X and there exists $0 < \delta < \varepsilon$ such that for each n

$$\sum_{\alpha=1}^{3} (\nabla_{e_\alpha} f_n)^2 \leq 1 - \delta \text{ almost everywhere on } X , \tag{36}$$

which means in particular that the hypersurfaces $\mathcal{S}_n$ are spacelike uniformly in $x \in X$ and n. For each n, we have

$$\|\Psi\|^2_{L^2(\mathcal{S}_n)} = \|\Psi\|^2_{L^2(X_0)}$$

and the L^2 norm of Ψ on $\mathcal{S}_n$ has the explicit expression

$$
\begin{aligned}
\|\Psi\|^2_{L^2(\mathcal{S}_n)} = \int_X \Big(& \nu_n^{AA'}(f_n(x), x) \phi_A(f_n(x), x) \bar{\phi}_{A'}(f_n(x), x) \\
& + \nu_n^{AA'}(f_n(x), x) \chi_{A'}(f_n(x), x) \bar{\chi}_A(f_n(x), x) \Big) \, \mathrm{d}\sigma_n(f_n(x), x) ,
\end{aligned}
$$

where $\mathrm{d}\sigma_n$ is the Leray form on $\mathcal{S}_n$ associated with the parametrization by f_n, i.e.

$$\mathrm{d}\sigma_n = \nu_n \lrcorner \mathrm{dVol}^4 ,$$

and ν_n^a is the future oriented unit normal vector field to $\mathcal{S}_n$. This has a simple expression in terms of f_n :

$$V_n = e_0 + \sum_{\alpha=1}^{3} \nabla_{e_\alpha} f_n \, e_\alpha , \quad \nu_n^a = \frac{V_n^a}{g_{bc} V_n^b V_n^c} .$$

The properties of f_n imply that the components of ν_n^a in the orthonormal frame are bounded in $L^\infty(X)$ and converge almost everywhere on X towards the components of ν^a. Hence by continuity of Ψ on $\mathcal{X}$ and by dominated convergence, it follows that

$$\|\Psi\|^2_{L^2(\mathcal{S}_n)} \to \|\Psi\|^2_{L^2(\mathcal{S})} \text{ as } n \to +\infty$$

and equality (34) is established.

We now follow the same strategy with $\nabla_{\mathcal{T}} \Psi$ instead of Ψ : it satisfies equation (4) with a perturbation which is a smooth potential ; this prevents us from obtaining the same equality, but we have equivalence of norms nevertheless using standard Gronwall-type arguments. Hence, there exists a constant $C > 1$ which is independent of Ψ and depends only on T_1 and T_2 such that

$$\frac{1}{C} \|\nabla_{\mathcal{T}} \Psi\|^2_{L^2(\mathcal{S})} \leq \|\nabla_{\mathcal{T}} \Psi\|^2_{L^2(X_0)} \leq C \|\nabla_{\mathcal{T}} \Psi\|^2_{L^2(\mathcal{S})} .$$

It remains to prove that $\|\nabla_{\mathcal{T}} \Psi\|^2_{L^2(\mathcal{S})} + \|\Psi\|^2_{L^2(\mathcal{S})}$ is equivalent to the squared $H^1(\mathcal{S})$ norm of Ψ. In order to establish this, we decompose the Dirac equation in terms of a derivative along $\mathcal{T}$

---

[3]For the existence of such a sequence of smooth hypersurfaces approaching $\mathcal{S}$, see [9], Lemma 3.

and derivatives tangent to $\mathcal{S}$. Such a calculation is most easily performed using the expression of equation (4) in the orthonormal basis $\{e_0, e_1, e_2, e_3\}$ (recall that $\mathcal{T} = \sqrt{2}e_0$) :

$$e_0 . \nabla_{e_0} \Psi + \sum_{\alpha=1}^{3} e_\alpha . \nabla_{e_\alpha} \Psi = \mathcal{P}\Psi \tag{37}$$

where $\mathcal{P}$ is a smooth potential (involving purely the mass and charge terms). This can be decomposed as follows :

$$\left[\left(e_0 - \sum_{\alpha=1}^{3} \nabla_{e_\alpha} f \, e_\alpha \right) . \nabla_{e_0} \Psi \right] = - \left[\sum_{\alpha=1}^{3} e_\alpha . \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \right] + \mathcal{P}\Psi \,.$$

Clifford multiplying by the vector

$$W = e_0 - \sum_{\alpha=1}^{3} \nabla_{e_\alpha} f \, e_\alpha \,,$$

we obtain

$$\left(1 - \sum_{\alpha=1}^{3} |\nabla_{e_\alpha} f|^2 \right) \nabla_{e_0} \Psi = -W . \left[\sum_{\alpha=1}^{3} e_\alpha . \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \right] + W . \mathcal{P}\Psi \,,$$

or equivalently

$$\nabla_{\mathcal{T}} \Psi = \frac{\sqrt{2}}{1 - \sum_{\alpha=1}^{3} |\nabla_{e_\alpha} f|^2} W . \left(- \left[\sum_{\alpha=1}^{3} e_\alpha . \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \right] + \mathcal{P}\Psi \right) . \tag{38}$$

Remark 5.2. *The vector W is obtained from the vector V by reversing the signs of all the spacelike components. One can infer from the expression of the Clifford product in terms of 2-spinor indices given in [11] that Clifford multiplication of a spinor Ξ by W is equivalent to the contraction of the 2-spinor parts of Ξ with the spinor form $V^{AA'}$ of the normal vector V. If the hypersurface $\mathcal{S}$ were null, then Clifford multiplying equation (4) by W (or equivalently contracting $V^{AA'}$ into it) would give us the part of (4) tangent to $\mathcal{S}$.*

Since $\mathcal{S}$ is uniformly spacelike, $1 - \sum_{\alpha=1}^{3} \left(\nabla_{e_\alpha} f \right)^2$ is bounded and bounded away from zero uniformly (or more precisely essentially uniformly since we are dealing with almost everywhere defined functions) on $\mathcal{S}$. Moreover,

$$g_{ab} V^a V^b = g_{ab} W^a W^b = 1 - \sum_{\alpha=1}^{3} \left(\nabla_{e_\alpha} f \right)^2 \,,$$

i.e. W is essentially uniformly timelike on $\mathcal{S}$ just like V. Hence the Clifford multiplication by W is an isomorphism at each point where f is differentiable, with essential uniform bounds on $\mathcal{S}$ for its norm and the norm of its inverse.

Let us now calculate the norm at each point of $\mathcal{S}$ of the spinor

$$\sum_{\alpha=1}^{3} e_\alpha . \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \,.$$

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In order to simplify notations, we denote

$$\Psi_{|\alpha} := \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \, .$$

We have

$$\left| \sum_{\alpha=1}^3 e_\alpha . \Psi_{|\alpha} \right|^2 \;=\; \left(e_0 . \sum_{\alpha=1}^3 e_\alpha . \Psi_{|\alpha} \, , \; \sum_{\beta=1}^3 e_\beta . \Psi_{|\beta} \right)$$

$$=\; \sum_{\alpha,\beta=1}^3 \left(e_0 . e_\alpha . \Psi_{|\alpha} \, , \; e_\beta . \Psi_{|\beta} \right) \, .$$

The terms with $\alpha = \beta$ give

$$\left(e_0 . e_\alpha . \Psi_{|\alpha} \, , \; e_\alpha . \Psi_{|\alpha} \right) \;=\; - \left(e_0 . e_\alpha . e_\alpha . \Psi_{|\alpha} \, , \; \Psi_{|\alpha} \right)$$

$$=\; \left(e_0 . \Psi_{|\alpha} \, , \; \Psi_{|\alpha} \right)$$

$$=\; \left| \Psi_{|\alpha} \right|^2 \, .$$

For $\alpha \neq \beta$, the terms in the sum cancel one another two by two since

$$\left(e_0 . e_\alpha . \Psi_{|\alpha} \, , \; e_\beta . \Psi_{|\beta} \right) \;=\; - \left(e_\beta . \Psi_{|\beta} \, , \; e_0 . e_\alpha . \Psi_{|\alpha} \right)$$

$$=\; - \left(e_0 . e_\beta . \Psi_{|\beta} \, , \; e_\alpha . \Psi_{|\alpha} \right) \, .$$

It follows that

$$\left\| \sum_{\alpha=1}^3 e_\alpha . \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \right\|^2_{L^2(\mathcal{S})} \, .$$

is equivalent to

$$\sum_{\alpha=1}^3 \left\| \left(\nabla_{e_\alpha} + \nabla_{e_\alpha} f \, \nabla_{e_0} \right) \Psi \right\|^2_{L^2(\mathcal{S})} = \sum_{\alpha=1}^3 \left\| \nabla_{\tau_\alpha} \Psi \right\|^2_{L^2(\mathcal{S})} \, .$$

Hence, from (38) and the definition of the norm in $H^1(\mathcal{S})$, we conclude that

$$\| \Psi \|^2_{H^1(\mathcal{S})} \lesssim \| \nabla_{\mathcal{T}} \Psi \|^2_{L^2(\mathcal{S})} + \| \Psi \|^2_{L^2(\mathcal{S})} \lesssim \| \Psi \|^2_{H^1(\mathcal{S})}$$

with constants in the estimates depending only on the Lipschitz norm of f on X. This concludes the proof of Lemma 5.1. $\qquad \square$

The first consequence of this lemma is the existence of a trace on $\mathcal{S}$ for minimum regularity solutions :

Corollary 5.1. *The trace operator*

$$\Gamma \,:\, \mathcal{C}^\infty(X_0 \,;\, \mathbb{S}_A \oplus \mathbb{S}^{A'}) \longrightarrow L^2(\mathcal{S} \,;\, \mathbb{S}_A \oplus \mathbb{S}^{A'}) \,,$$

which to smooth data Φ on X_0 associates the trace on $\mathcal{S}$ of the smooth solution Ψ of (4) such that $\Psi|_{X_0} = \Phi$, extends as a continuous linear map still denoted Γ :

$$\Gamma \,:\, L^2(X_0 \,;\, \mathbb{S}_A \oplus \mathbb{S}^{A'}) \longrightarrow L^2(\mathcal{S} \,;\, \mathbb{S}_A \oplus \mathbb{S}^{A'}) \,.$$

Moreover, Γ satisfies for all $\Phi \in L^2(X_0 \,;\, \mathbb{S}_A \oplus \mathbb{S}^{A'})$,

$$\| \Gamma \Phi \|^2_{L^2(\mathcal{S})} = \| \Phi \|^2_{L^2(X_0)} \,.$$

This entails that Γ is one-to-one and with closed range.

The restriction of Γ to $H^1(X_0\,;\,\mathbb{S}_A\oplus\mathbb{S}^{A'})$ is continuous from this space to $H^1(\mathcal{S}\,;\,\mathbb{S}_A\oplus\mathbb{S}^{A'})$, and satisfies

$$C_1\|\Gamma\Phi\|^2_{H^1(\mathcal{S})}\le\|\Phi\|^2_{H^1(X_0)}\le C_2\|\Gamma\Phi\|^2_{H^1(\mathcal{S})}\,,$$

where C_1 and C_2 are the constants appearing in (35).

Remark 5.3. *Note that in the case of H^1 data, the solution is in $H^1_{\text{loc}}(\mathcal{X})$ and Γ is therefore a trace in the usual sense.*

All that we now need to conclude the proof of the theorem is to show that for data $\Phi\in H^1(\mathcal{S})$, we can construct a solution Ψ whose trace on $\mathcal{S}$ is Φ, i.e. that the range of Γ contains $H^1(\mathcal{S})$ (which is dense in $L^2(\mathcal{S})$). Using again the surfaces $\mathcal{S}_n$ and the functions f_n defined at the beginning of the proof of Lemma 5.1, we consider some data $\Phi\in H^1(\mathcal{S}\,;\,\mathbb{S}_A\oplus\mathbb{S}^{A'})$ and we push them along the flow of the vector field $\mathcal{T}$ as data Φ_n on $\mathcal{S}_n$. Since the sequence $\{f_n\}_n$ is bounded in $W^{1,\infty}(X)$, not only is each Φ_n in $H^1(\mathcal{S}_n\,;\,\mathbb{S}_A\oplus\mathbb{S}^{A'})$, but the norm

$$\|\Phi_n\|_{H^1(\mathcal{S}_n\,;\,\mathbb{S}_A\oplus\mathbb{S}^{A'})}$$

is bounded in n. By standard theorems, for each n, there exists a unique solution

$$\Psi_n\in\mathcal{C}(\mathbb{R}_t\,;\,H^1(X))\cap\mathcal{C}^1(\mathbb{R}_t\,;\,L^2(X))$$

of (4) such that $\Psi_n|_{\mathcal{S}_n}=\Phi_n$. Now by Lemma 5.1, the sequence Ψ_n is bounded in $\mathcal{C}(I\,;\,H^1(X))\cap\mathcal{C}^1(I\,;\,L^2(X))$ for any bounded time interval I containing 0 and such that $I\times X$ contains all hypersurfaces $\mathcal{S}_n$ and $\mathcal{S}$. Modulo the extraction of a subsequence, we can therefore assume that Ψ_n converges weakly in $H^1(I\times X)$ and in $H^1(X_0)$, towards a solution

$$\Psi\in\mathcal{C}(I\,;\,L^2(X))\,,$$

of equation (4) which naturally extends as

$$\Psi\in\mathcal{C}(\mathbb{R}_t\,;\,L^2(X))\,.$$

Since $\Psi(0)\in H^1(X)$, it follows that Ψ is more regular :

$$\Psi\in\mathcal{C}(\mathbb{R}_t\,;\,H^1(X))\cap\mathcal{C}^1(\mathbb{R}_t\,;\,L^2(X))\,.$$

Now, using the Rellich-Kondrachov compact embedding theorem, it follows that modulo the extraction of another subsequence, Ψ_n converges towards Ψ strongly in $H^{1/2}(I\times X)$, therefore by standard trace theorems, strongly in $L^2(\mathcal{S})$. It remains to prove that $\Gamma(\Psi(0))=\Phi$, or more simply that the trace of Ψ on $\mathcal{S}$ is equal to Φ. To establish this last result, we project spinors on a given global spin-frame, still denoting Ψ_n, Ψ, Φ_n and Φ the vectors of the components of the correponding spinors in the spin-frame. We have

$$\begin{aligned}\int_X|\Psi(f(x),x)-\Phi_n(f_n(x),x)|^2\mathrm{d}\mu&=\int_X|\Psi(f(x),x)-\Psi_n(f_n(x),x)|^2\mathrm{d}\mu\\&\le\int_X|\Psi(f(x),x)-\Psi_n(f(x),x)|^2\mathrm{d}\mu\\&\quad+\int_X|\Psi_n(f(x),x)-\Psi_n(f_n(x),x)|^2\mathrm{d}\mu\,.\end{aligned}$$

The first integral on the right-hand side tends to zero since $\Psi_n \to \Psi$ strongly in $L^2(\mathcal{S})$. As for the second, denoting $(f(x), f_n(x))$ the interval between $f(x)$ and $f_n(x)$,

$$
\begin{aligned}
\int_X |\Psi_n(f_n(x), x) - \Psi_n(f(x), x)|^2 \mathrm{d}\mu
&\leq \int_X |\int_{(f(x),f_n(x))} \partial_t \Psi_n(t, x) \mathrm{d}t|^2 \mathrm{d}\mu \\
&\leq \int_X |f_n(x) - f(x)| \int_{(f(x),f_n(x))} |\partial_t \Psi_n(t, x)|^2 \mathrm{d}t \mathrm{d}\mu \\
&\leq \sup_{x \in X} |f_n(x) - f(x)| \int_{I \times X} |\partial_t \Psi_n(t, x)|^2 \mathrm{d}t \mathrm{d}\mu.
\end{aligned}
$$

The factor in front of the integral tends to zero since f_n converges uniformly towards f on X and the integral is bounded since Ψ_n is bounded in $\mathcal{C}^1(I\,;\; L^2(X))$. It follows that

$$
\int_X |\Psi(f(x), x) - \Phi_n(f_n(x), x)|^2 \mathrm{d}\mu
$$

tends to zero. But since by construction $\Phi_n(f_n(x), x)$ tends to $\Phi(f(x), x)$ uniformly on X, this implies that the trace of Ψ on $\mathcal{S}$ is equal to Φ. The proof is complete. $\qquad\square$

5.2 Proof of Theorems 1 and 2

For these proofs, we assume that our coordinate system and Newman-Penrose tetrad are defined on a subdomain of Ω that is slightly larger than D, namely on $\mathcal{J}^+(\zeta(-\eta)) \cap \mathcal{J}^-(\zeta(2T + \eta))$ for some $\eta > 0$. This is always possible since D is compact inside the open set Ω.

First recall that we have $l = \frac{2}{N} \partial_v$, $n = \frac{2}{N} \partial_u$ (see lemma 3.1) and that m lies in the tangent planes to the 2-surfaces of constant u and v, which means that $m^a \partial_a$ involves only derivatives with respect to ω. Using (11) we see that the Dirac equation takes the form :

$$
\partial_t \Psi = i\tilde{H}\Psi\,;\; \tilde{H} = \gamma D_r + \tilde{P}_\omega + \tilde{Q}, \; \gamma = Diag(1, -1, -1, 1), \tag{39}
$$

where D_r denotes $-i\partial_r$. Here $\tilde{P}_\omega$ is a differential operator with derivatives only in the angular directions and $\tilde{Q}$ is a potential. Note that the operators $\tilde{P}_\omega$ and $\tilde{Q}$ depend on t.

Thanks to inequalities (30), we only need to prove the surjectivity of the trace operator, i.e. to solve the characteristic Cauchy problem for (4) with characteristic data in $\mathcal{H}_{\mathcal{C}_0^+}$. We consider two scalar functions $g_1(r, \omega)$ and $g_4(r, \omega)$ on Σ_T and we extend them as constant functions on the integral lines of ∂_t. From now on, we will always identify a function g on Σ_T with its extension that we still denote by g. In particular, we shall consider such functions as functions on Σ_T or on $\mathcal{C}_0^+$ according to convenience.

For $g_{1,4} \in \mathcal{H}_{\mathcal{C}_0^+}$, we wish to find a solution Ψ, whose trace on Σ_T is in $H^1(\Sigma_T)$, of the characteristic Cauchy problem :

$$
\begin{cases}
\partial_t \Psi = i\tilde{H}\Psi \text{ on } \mathcal{X}, \\
\Psi_{1,4}(r, r, \omega) = g_{1,4}(r, \omega), \; (r, \omega) \in [0, T] \times S^2.
\end{cases} \tag{40}
$$

We first assume that $g_{1,4} \in \mathcal{C}^\infty(\Sigma_T)$ and define $g_{2,3}$ by

$$
\begin{pmatrix} g_2|_{\mathcal{C}_0^+} \\ g_3|_{\mathcal{C}_0^+} \end{pmatrix}(v, \omega) := \left(K \begin{pmatrix} g_1|_{\mathcal{C}_0^+} \\ g_4|_{\mathcal{C}_0^+} \end{pmatrix} \right)(v, \omega),
$$

$$
\begin{pmatrix} g_2|_{\mathcal{C}_0^+} \\ g_3|_{\mathcal{C}_0^+} \end{pmatrix}(0, \omega) = \begin{pmatrix} -ie^{-i\theta(\omega)/2}\, g_1|_{\mathcal{C}_0^+} \\ -ie^{i\theta(\omega)/2}\, g_4|_{\mathcal{C}_0^+} \end{pmatrix}(0, \omega').
$$

Let us now open the cone by a factor $0 < \lambda < 1$, $|\lambda - 1| << 1$. The new cone becomes spacelike and we can solve the corresponding Cauchy problem by the previous results. We recover the solution of the Goursat problem in the limit $\lambda \to 1$. More precisely, for $\lambda < 1$, we extend $g_{1,4}$ to smooth functions on

$$\Sigma_T^\lambda := \{t = T, \ r \in [0, T/\lambda], \ \omega \in S^2\}$$

and thus $g_{2,3}$ to functions in $H^1(\Sigma_T^\lambda)^4$ and we consider the Cauchy problem :

$$\begin{cases} \partial_t \Psi^\lambda &=& i\tilde{H}\Psi^\lambda, \\ \Psi^\lambda(\lambda r, r, \omega) &=& g(r, \omega); \ (r, \omega) \in [0, \frac{T}{\lambda}] \times S^2. \end{cases} \tag{41}$$

We put $\mathcal{C}_0^{+,\lambda} = \{(\lambda r, r, \omega); 0 \le r \le T/\lambda, \omega \in S^2\}$. The tangent plane to $\mathcal{C}_0^{+,\lambda}$ at a given point p is given for $r \ne 0$ by

$$\begin{aligned} T_p \mathcal{C}_0^{+,\lambda} &=& Span\{\lambda \partial_t + \partial_r, \partial_\omega\} \\ &=& Span\{(1 + \lambda)l + (\lambda - 1)n, \partial_\omega\}. \end{aligned}$$

Therefore

$$l_\lambda = \frac{1}{2}(1 + \frac{1}{\lambda})l + \frac{1}{2}(\frac{1}{\lambda} - 1)n$$

is orthogonal to $\mathcal{C}_0^{+,\lambda}$ and

$$n_\lambda = \frac{1}{2}(1 + \lambda)n + \frac{1}{2}(\lambda - 1)l$$

is transverse to $\mathcal{C}_0^{+,\lambda}$. We also have :

$$g(l_\lambda, n_\lambda) = 1.$$

Therefore we obtain :

$$\int_{\mathcal{C}_0^{+,\lambda}} *(\phi_A \bar{\phi}_{A'} \mathrm{d}x^{AA'} + \chi_{A'} \bar{\chi}_A \mathrm{d}x^{AA'}) = \int_{\mathcal{C}_0^{+,\lambda}} \left(\frac{1 + \lambda}{2\lambda} |\Psi_{1,4}|^2 + \frac{1 - \lambda}{2\lambda} |\Psi_{2,3}|^2 \right) \mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}}$$

with $\mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}} = n_\lambda \lrcorner \mathrm{dVol}^4$. The equation (41) has by Theorem 3 a unique solution and we have the estimate (see (34)) :

$$\int_{[0,T] \times S^2} |\Psi^\lambda|^2(T, r, \omega) \mathrm{d}\sigma_{\Sigma_T} = \int_{\mathcal{C}_0^{+,\lambda}} \left(\frac{1 + \lambda}{2\lambda} |g_{1,4}|^2(r, \omega) + \frac{1 - \lambda}{2\lambda} |g_{2,3}|^2 \right) \mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}}.$$

Now $\Phi^\lambda = \tilde{H}\Psi^\lambda$ is solution of

$$\partial_t \Phi^\lambda = i\tilde{H}\Phi^\lambda + [\partial_t, i\tilde{H}]\Psi^\lambda$$

and we have the estimate (see (35)) :

$$\begin{aligned} &\int_{[0,T] \times S^2} |\Phi^\lambda|^2(T, r, \omega) \mathrm{d}\sigma_{\Sigma_T} \\ &\lesssim \int_{\mathcal{C}_0^{+,\lambda}} \left(\frac{1 + \lambda}{2\lambda} |\Phi_{1,4}^\lambda|^2(\lambda r, r, \omega) + \frac{1 - \lambda}{2\lambda} |\Phi_{2,3}^\lambda|^2(\lambda r, r, \omega) \right) \mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}} \\ &+ \int_{\mathcal{C}_0^{+,\lambda}} \left(\frac{1 + \lambda}{2\lambda} |\Psi_{1,4}^\lambda|^2(\lambda r, r, \omega) + \frac{1 - \lambda}{2\lambda} |\Psi_{2,3}^\lambda|^2(\lambda r, r, \omega) \right) \mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}}. \end{aligned}$$

---

[4]In fact, under the action of K followed by the projection on Σ_T^λ along the integral lines of ∂_t, we get functions which are smooth on Σ_T^λ except at $r = 0$ where they are Lipschitz, so we have more regularity than H^1 but H^1 is all we need for applying Theorem 3.

Therefore we have to calculate $\Phi^\lambda(\lambda r, r, \omega)$. To this purpose we introduce the following coordinates :

$$\left.\begin{array}{rcl} \tau &=& t - \lambda r, \\ x &=& r \end{array}\right\} \Rightarrow \partial_t = \partial_\tau \,;\, \partial_r = \partial_x - \lambda \partial_\tau .$$

We have

$$\begin{aligned} \partial_t \Psi^\lambda &=& i\tilde{H}\Psi^\lambda \\ \Leftrightarrow \partial_\tau \Psi^\lambda &=& (1+\gamma\lambda)^{-1}\left(\gamma\partial_x \Psi^\lambda + i(\tilde{P}_\omega + \tilde{Q})\Psi^\lambda\right). \end{aligned} \tag{42}$$

Using (42) we calculate :

$$\begin{aligned} \tilde{H}\Psi^\lambda &=& Diag\left(\frac{1}{1+\lambda},\frac{1}{\lambda-1},\frac{1}{\lambda-1},\frac{1}{1+\lambda}\right) D_x \Psi^\lambda \\ &+& Diag\left(\frac{1}{1+\lambda},\frac{1}{1-\lambda},\frac{1}{1-\lambda},\frac{1}{1+\lambda}\right)\left(\tilde{P}_\omega + \tilde{Q}\right)\Psi^\lambda . \end{aligned}$$

Recalling that $g(r,\omega) = \Psi^\lambda(\lambda r, r, \omega)$ we find :

$$\begin{aligned} \Phi^\lambda(\lambda r, r, \omega) &=& Diag\left(\frac{1}{1+\lambda},\frac{1}{\lambda-1},\frac{1}{\lambda-1},\frac{1}{1+\lambda}\right) D_r g(r,\omega) \\ &+& Diag\left(\frac{1}{1+\lambda},\frac{1}{1-\lambda},\frac{1}{1-\lambda},\frac{1}{1+\lambda}\right)\left(\tilde{P}_\omega + \tilde{Q}\right) g(r,\omega) \\ &=:& g_H^\lambda . \end{aligned} \tag{43}$$

We therefore obtain the estimate :

$$\begin{aligned} \int_{[0,T]\times S^2} &|\Phi^\lambda|^2(T,r,\omega)\mathrm{d}\sigma_{\Sigma_T} \\ &\lesssim \int_{\mathcal{C}_0^{+,\lambda}}\left(\frac{1+\lambda}{2\lambda}|g_{1,4}|^2(r,\omega) + \frac{1-\lambda}{2\lambda}|g_{2,3}|^2(r,\omega)\right)\mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}} \\ &+ \int_{\mathcal{C}_0^{+,\lambda}}\left(\frac{1+\lambda}{2\lambda}|(g_H^\lambda)_{1,4}|^2(r,\omega) + \frac{1-\lambda}{2\lambda}|(g_H^\lambda)_{2,3}|^2(\lambda r, r, \omega)\right)\mathrm{d}\sigma_{\mathcal{C}_0^{+,\lambda}}. \end{aligned} \tag{44}$$

Now recall that g satisfies the transport equations along the cone :

$$\partial_r g_{2,3} = i((\tilde{P}_\omega + \tilde{Q})g)_{2,3}.$$

It follows

$$(g_H^\lambda)_{2,3} = 0.$$

and the R.H.S of (44) is uniformly bounded in $|\lambda - 1| << 1$:

$$\|\Psi^\lambda(T,.)\|_{H^1(\Sigma_T)} \lesssim 1.$$

Repeating the above arguments for the spaces $H^1(\Sigma_t)$ we see that we can extract a subsequence, still denoted Ψ^λ, s.t.

$$\begin{aligned} \Psi^\lambda &\rightharpoonup \Psi & H^1(\Sigma_T), \\ \Psi^\lambda &\rightharpoonup \Psi & H^1(D_T), \\ \Psi^\lambda &\rightarrow \Psi & H^{1/2}(D_T). \end{aligned}$$

Ψ is a solution of the Dirac equation and we have :

$$||\Psi||_{H^1(\Sigma_T)} \lesssim ||g||_{\mathcal{H}_{\mathcal{C}_0^+}}, \ ||\Psi||_{H^1(D_T)} \lesssim ||g||_{\mathcal{H}_{\mathcal{C}_0^+}}. \tag{45}$$

We want to check that

$$\Psi_{1,4}(r,r,\omega) \ = \ g_{1,4}(r,\omega) \quad \forall 0 \le r \le T.$$

In fact we can even show :

$$\Psi(r,r,\omega) = g(r,\omega).$$

We estimate :

$$\int_0^T \int_{S^2} |g(r,\omega) - \Psi^\lambda(r,r,\omega)|^2 \mathrm{d}\sigma_{\mathcal{C}_0^+}$$
$$= \int_0^T \int_{S^2} |\Psi^\lambda(\lambda r, r, \omega) - \Psi^\lambda(r,r,\omega)|^2 \mathrm{d}\sigma_{\mathcal{C}_0^+}$$
$$= \int_0^T \int_{S^2} \left| \int_{\lambda r}^r \partial_t \Psi^\lambda(t,r,\omega) \mathrm{d}t \right|^2 \mathrm{d}\sigma_{\mathcal{C}_0^+}$$
$$\le |\lambda - 1|T \int_0^T \int_{S^2} \int_{\lambda r}^r |\partial_t \Psi^\lambda(t,r,\omega)|^2 \mathrm{d}t \, \mathrm{d}\sigma_{\mathcal{C}_0^+}$$
$$\lesssim T^2 |\lambda - 1| ||\tilde{H}\Psi^\lambda||^2_{L^2(\Sigma_T^\lambda)} \lesssim T^2 |\lambda - 1| \to 0.$$

On the other hand :

$$\int_0^T \int_{S^2} |\Psi^\lambda(r,r,\omega) - \Psi(r,r,\omega)|^2 \mathrm{d}\sigma_{\mathcal{C}_0^+} \le ||\Psi^\lambda - \Psi||^2_{H^{1/2}(K_T)} \to 0.$$

Thus $\Psi(r,r,\omega) = g(r,\omega)$.

If $g_{1,4} \in \mathcal{H}_{\mathcal{C}_0^+}$ we approach it by a sequence $g_{1,4}^n$ of smooth data (as viewed on Σ_T) and the corresponding solutions converge to a solution Ψ. The trace of Ψ on $\mathcal{C}_0^+$ exists and we have :

$$\int_{[0,T]\times S^2} |\Psi_{1,4}(r,r,\omega) - g_{1,4}(r,\omega)|^2 \mathrm{d}\sigma_{\mathcal{C}_0^+} \lesssim ||\Psi - \Psi^n||^2_{H^{1/2}(K_T)} + \int_{[0,T]\times S^2} |(g^n - g)_{1,4}|^2 \mathrm{d}\sigma_{\mathcal{C}_0^+} \to 0$$

and thus

$$\Psi_{1,4}(r,r,\omega) = g_{1,4}(r,\omega).$$

If $g_{1,4} \in L^2$ we again approach it by a sequence of smooth data $g_{1,4}^n$. The corresponding solutions are in $H^1(\Sigma_T)$ and converge to some Ψ in $L^2(\Sigma_T)$. By definition of the extension of the trace, we have $\Gamma\Psi = g_{1,4}$. This concludes the proof of Theorems 1 and 2. $\qquad\square$

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